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On the radial filling of a rotating cylinder

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Some aspects of the radial filling of a finite rotating cylinder are considered in the limit of a small Ekman number E. Three main cases are distinguished by the value of τ_F, the ratio of filling and spin-up times. When $\tau_F \ll 1$ the effect of the Ekman layers is unimportant and new fluid accumulates behind that already contained by an essentially radial flux. For $\tau_F \sim 1$ the Ekman layers are active and entering fluid is added both ahead and behind the initially contained fluid core, which undergoes a process similar to spin-up with the notable difference that here the Ekman layers are non-divergent. In both cases the Rossby number ϵ is $O(1)$. When $\tau_F \gg 1$, ϵ is small and the Ekman layers control the (quasi-steady) filling. The new fluid is then transported through boundary layers and spread on the moving front from the inside throughout an $E^{\frac{1}{3}}$ layer imbedded in a weak $E^{\frac{1}{4}}$ layer.

1. Introduction

The radial filling of a rotating container is of interest in separation and propulsion processes. The closely related steady-state source–sink flow has been extensively studied (e.g. Lewellen 1965; Barcilon 1966; Hide 1968; Bennetts & Hocking 1973; Bennetts & Jackson 1974; Conlisk & Walker 1981); the radial filling process has been considered by Hocking (1970), who analysed a rapid process in which the Ekman layers' contribution is insignificant.

This paper examines the function of the Ekman layers in the filling process; especially the slow filling at a small Rossby number, which is dominated by these layers. The equations of motion for the inviscid interior are formulated in §2. Their subsequent solution clarifies general features of the process under investigation. In particular, it points out the differences between 'rapid', 'moderate' and 'slow' filling corresponding to small, $O(1)$ and large values of τ_F (the ratio of filling and spin-up times) and establishes the relation between the Rossby number of the flow ϵ and τ_F. The analysis of the flow field for small ϵ is extended in §3 to account for 'vertical' shear layers on the liquid–gas (or vacuum) interface, which are essential to the process.

2. Analysis of general features

Consider an annulus of height H^*, inner and outer radii r_I^* and r_O^*, which rotates with angular velocity Ω^* around its axis of symmetry (figure 1). (The upper asterisk designates dimensional variables.) An incompressible fluid initially occupies the annular region $a^* \leqslant r^* \leqslant r_O^*$ and is in solid rotation. Starting at $t^* = 0$ a volume flux of constant rate $-\dot{Q}^*$ is applied uniformly on the outer permeable wall. The fluid that enters the container has the azimuthal velocity of the outer wall. It is assumed that $\Omega^{*2} r_I^*$ is large enough for gravitational effects to be negligible and for the Ekman

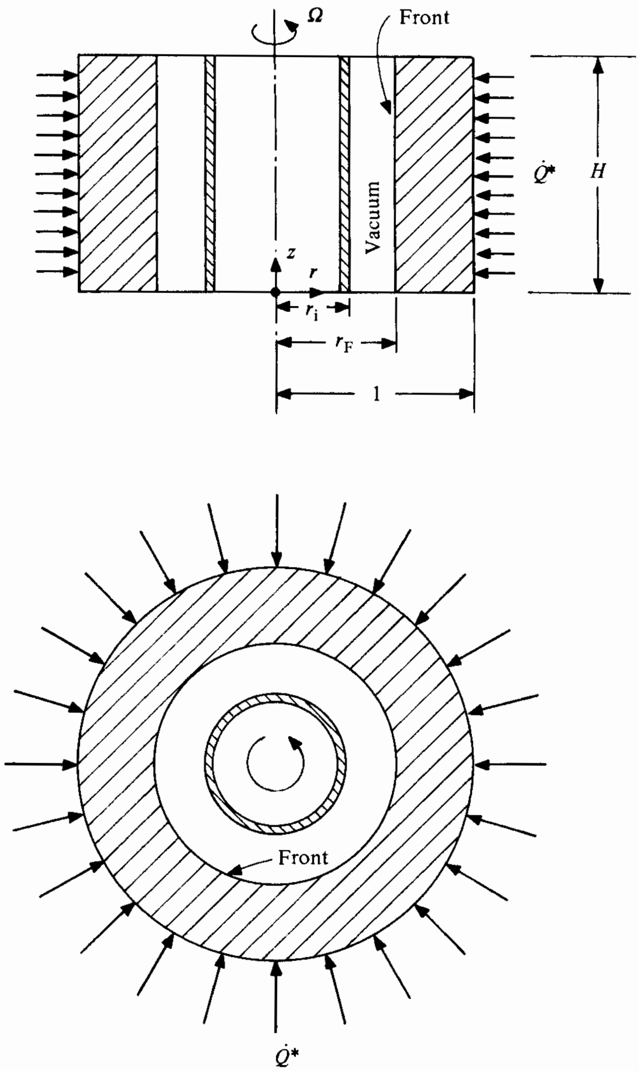

FIGURE 1. Schematic section of the rotating container filling through the outer wall. Initially $r_F = a$.

number $E = \nu^*/\Omega^* r_O^{*2}$ to be small (where ν^* is the kinematic viscosity). The object is to describe the time-dependent fluid motion for $t^* > 0$.

The methods used are essentially similar to those employed in the study of the nonlinear spin-up problem (Wedemeyer 1964; Greenspan 1968; a topic reviewed by Benton & Clark 1974). The flow field consists of thin 'horizontal' boundary layers of Ekman type and a nearly inviscid interior motion. We shall consider in detail only the solution in the interior. Let U, V and W be the radial, azimuthal and axial interior velocity components in an *inertial* cylindrical coordinate system and scale the variables by the length r_O^*, time Ω^{*-1} and velocity $\Omega^* r_O^*$. Since the volume flux in the Ekman layers is $O(E^{\frac{1}{2}})$, the velocity components in the interior are assumed to be expressible as an asymptotic series in powers of $E^{\frac{1}{2}}$. Upon substitution in the equations of motion, the leading terms of U and V in this expansion are found to be independent of the axial coordinate z (Greenspan 1968, §3.7). Consequently, the azimuthal momentum equation is

$$\frac{\partial}{\partial t} V(r,t) + U(r,t)\frac{1}{r}\frac{\partial}{\partial r} rV(r,t) = 0. \tag{2.1}$$

The main matching condition of the interior and boundary-layer flows arises from the conservation of volume flux:

$$2\tilde{Q} + \int_0^H 2\pi r U(r,t)\,\mathrm{d}z = -\dot{Q}. \tag{2.2}$$

Here $\dot{Q}$ is the volume flow rate in one Ekman layer, which will be approximated by

$$\tilde{Q} = -\pi E^{\frac{1}{2}} r[V(r,t) - r]. \tag{2.3}$$

This formula for the Ekman-layer 'suction', introduced by Wedemeyer (1964) (who actually used a slightly different coefficient of proportionality) can be regarded as a result of a momentum-integral analysis. Its validity can be proven for small-Rossby-number flows (i.e. $V/r \rightarrow 1$), but in general its applicability is *formally* restricted to special circumstances (see e.g. Hyun *et al.* 1983, where other important references are given). However, the use of (2.3) is vindicated by its simplicity, and a great deal of evidence that this approximation indeed reproduces the main features of the Ekman-layer suction in quite general flows. The utility of (2.3) is obvious in combining (2.1)–(2.3), the result of which is an equation for V alone:

$$H\frac{\partial V}{\partial t} + \left[E^{\frac{1}{2}}(V-r) - \frac{\dot{Q}}{2\pi r}\right]\left(\frac{1}{r}\frac{\partial}{\partial r} rV\right) = 0. \tag{2.4}$$

Upon introducing the scaled spin-up time $\tau = tE^{\frac{1}{2}}/H$, the specific angular momentum (or circulation) $\Gamma = rV$, and $\dot{q} = \dot{Q}/2\pi E^{\frac{1}{2}}$, (2.4) can be written as

$$\frac{\partial \Gamma}{\partial \tau} + \frac{1}{r}[(\Gamma - \dot{q}) - r^2]\frac{\partial \Gamma}{\partial r} = 0. \tag{2.5}$$

It can be anticipated that the magnitude of $\dot{q}$ has important implications for the flow-field properties. However, at this stage $\dot{q}$ is considered to be an $O(1)$ parameter. The initial and boundary conditions for Γ are

$$\left.\begin{aligned} \Gamma &= r^2 \quad (a \leqslant r \leqslant 1;\ \tau = 0), \\ \Gamma &= 1 \quad (\tau \geqslant 0;\ r = 1). \end{aligned}\right\} \tag{2.6}$$

The solution of (2.5)–(2.6) by the method of characteristics yields

$$\Gamma = r_*^2 \quad \text{on} \quad r^2 = r_*^2 - \dot{q}(1-\mathrm{e}^{-2\tau}) \quad (a \leqslant r_* \leqslant 1), \tag{2.7a}$$

$$\Gamma = 1 \quad \text{on} \quad r^2 = 1 - \dot{q}[1-\mathrm{e}^{-2(\tau-\gamma)}] \quad (\tau \geqslant \gamma \geqslant 0). \tag{2.7b}$$

Here r_* and γ are parameters which can be eliminated to obtain an explicit formula for $\Gamma(r,t)$, namely

$$\Gamma(r,\tau) = r^2 + \dot{q}(1-\mathrm{e}^{-2\tau}) \quad (R_{\mathrm{i}}(\tau) \leqslant r \leqslant R_{\mathrm{o}}(\tau)), \tag{2.8a}$$

$$\Gamma = 1 \quad (R_{\mathrm{o}}(\tau) < r \leqslant 1), \tag{2.8b}$$

where

$$R_{\mathrm{o}}^2(\tau) = 1 - \dot{q}(1-\mathrm{e}^{-2\tau}),$$

$$R_{\mathrm{i}}^2(\tau) = a^2 - \dot{q}(1-\mathrm{e}^{-2\tau}).$$

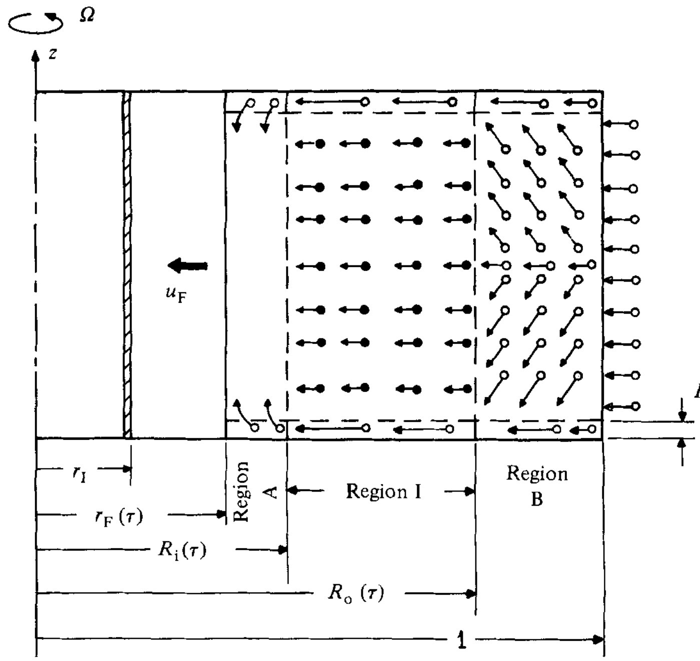

FIGURE 2. Schematic description of the instantaneous motion of fluid particles and the main interior regions: I (initially contained); A (ahead), B (behind); ○, new particles; ●, initially contained particles.

Using (2.8) and (2.2)–(2.3), the radial velocity is found to be

$$\left.\begin{aligned} U(r,\tau) &= -\frac{E^{\frac{1}{2}}}{Hr}\dot{q}\,\mathrm{e}^{-2\tau} = -\frac{\dot{Q}}{2\pi Hr}\mathrm{e}^{-2\tau} \quad (R_i(\tau) \leqslant r \leqslant R_0(\tau)),\\ U(r,\tau) &= -\frac{E^{\frac{1}{2}}}{Hr}[\dot{q}-(1-r^2)] \quad (R_0(\tau) \leqslant r \leqslant 1). \end{aligned}\right\} \tag{2.9}$$

The axial velocity is determined from mass continuity:

$$\left.\begin{aligned} W &= 0 \quad (R_i(t) \leqslant r < R_0(\tau)),\\ W &= E^{\frac{1}{2}}\left[\frac{2z}{H}-1\right] \quad (R_0(\tau) < r \leqslant 1). \end{aligned}\right\} \tag{2.10}$$

By simple time integration of (2.9) we find that $R_i(\tau)$ and $R_0(\tau)$ are the loci of the inner and outer rings of the initial fluid core. Hence the interesting consequence of (2.10) is that the Ekman layers adjacent to this core are non-divergent. This means that the new fluid will accumulate ahead of and behind the (moving) bulk of whatever fluid is initially present. Fluid cannot be added to the original mass. However, the foregoing solution does not describe the motion of all the fluid in the interior. Since the locus of the fluid front (in fact, a gas–liquid interface, which is assumed to be cylindrical) is

$$r_F^2(\tau) = a^2 - 2\dot{q}\tau, \tag{2.11}$$

it becomes clear that the flow in the region $r_F \leqslant r \leqslant R_i$ requires a special analysis. The width of this region increases with time because

$$R_i^2(\tau) - r_F^2(\tau) = \dot{q}(2\tau - \mathrm{e}^{-2\tau}).$$

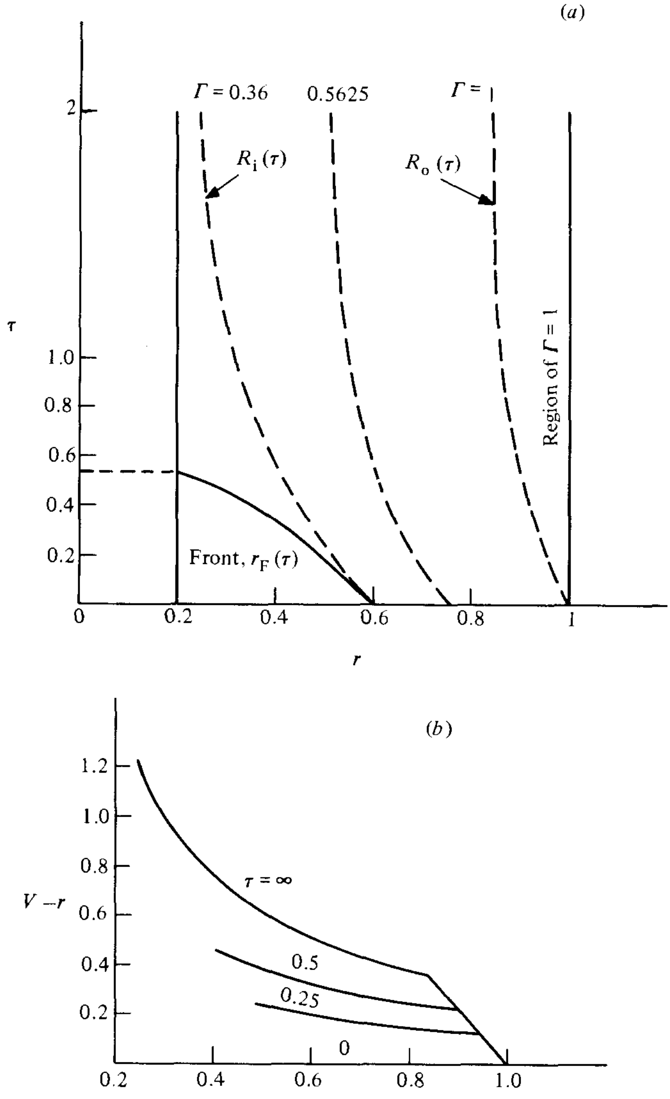

FIGURE 3. Typical solution (regions I and B) for a cylinder filled at $\dot{q} = 0.3$, inner radius $r_1 = 0.2$, front initially at $a = 0.6$. (A sink on the inner wall is assumed for $\tau > \tau_F$.) (*a*) Motion of fluid rings (conserving angular momentum) and of the front. (*b*) Surplus of local azimuthal velocity (over solid rotation) *vs.* r at different times.

Thus the interior flow can be divided in three main regions shown in figure 2: region I, $R_i(\tau) \leqslant r \leqslant R_o(\tau)$, of initially contained fluid is imbedded in regions B, $R_o(\tau) < r \leqslant 1$, and A, $r_F(\tau) \leqslant r < R_i(\tau)$, of new fluid.

Equations (2.9) and (2.10) show that region B contains fluid passing directly through the interior by a radial flux, while the fluid added to region A arrives via the Ekman layers. Region I has a passive role, and in the particular case of an initially empty container (i.e. $a = 1$) it reduces to the thin interface between the fluid particles filled through the interior and through the Ekman layers.

A typical solution for regions I and B is presented in figure 3. We note that in order to validate this solution for $\tau > \tau_F$ (τ_F is the filling time, when the front reaches the inner radius) an appropriate sink should be assumed on the inner boundary. Otherwise, the throughflow stops at $\tau = \tau_F$ and a spin-down process (not discussed here) takes place.

Viscous forces, which are neglected in this analysis, are expected to dominate the flow around the boundaries of regions I, A and B. In particular, according to (2.8) the derivative of V is discontinuous at $R_0(\tau)$ (see also figure 3*b*), and the corresponding free shear layer is discussed in the Appendix.

The physical interpretation of these results is as follows. A ring of fluid initially at $r = r_*$ is convected inwards and conserves its angular momentum. Because the fluid is transported to smaller radii, the azimuthal velocity will be everywhere larger than that of rigid rotation. The Ekman boundary layers, which are required to adjust the azimuthal velocity of the fluid to the rigid rotation of the 'horizontal' walls, give rise to a radially inward volume flux. Fluid is eventually expelled from these boundary layers into region A. The space between the last ring (originally at $r = 1$) and the outer wall (region B in figure 2) is filled with fluid convected from the wall with $\Gamma = 1$. The inward speed of the rings in region I, which is initially $O(E^{\frac{1}{2}}/H)$, decays exponentially on the spin-up timescale. The final position at $\tau = \infty$ of a ring originally at r_* is

$$r^2 = r_*^2 - \dot{q} \quad \text{for any } r_*, \quad a \leqslant r_* \leqslant 1; \tag{2.12}$$

and the azimuthal velocity in this case is then

$$V(r) = r + \dot{q}\frac{1}{r} \quad (a^2 - \dot{q} < r^2 < 1 - \dot{q}). \tag{2.13}$$

This value of $V(r)$ corresponds to the steady-state solution of a source–sink flow, in which the volume flux is transported entirely by the Ekman layers. Thus in the spin-up time every ring of initially contained fluid moves to the position where, owing to angular-momentum conservation, its new azimuthal velocity is that required for the entire imposed volume flux $-\dot{Q}^*$ to be pumped through the Ekman boundary layers. This is similar to the classical spin-up process, with the notable difference that here no fluid is sucked into the Ekman boundary layers, cf. (2.10). (On the other hand, since the filling process may be fairly rapid, some care is required in the physical interpretation of (2.12) and (2.13).)

For further investigation it is convenient to distinguish between 'rapid', 'moderate' and 'slow' filling corresponding to $\tau_F \ll 1, \tau_F \sim 1$ and $\tau_F \gg 1$, where τ_F is the filling time on the spin-up scale

$$\tau_F = \frac{1}{2\dot{q}}(a^2 - r_I^2). \tag{2.14}$$

In the case of 'rapid' filling, a Taylor-series expansion of (2.8) and (2.11) yields $r_F - R_i = O(\tau_F^2)$ and

$$V(r,\tau) - r \approx \dot{q}\frac{2\tau}{r} \quad (\tau \leqslant \tau_F \ll 1), \tag{2.15}$$

which, in view of (2.3), shows $\tilde{\dot{Q}}/\dot{Q} = O(\tau_F)$. We conclude that the contribution of the Ekman layers is negligible and the added fluid accumulates mainly behind that already inside. The flow field consists of regions I and B (figure 2); region A is negligibly small. This is exactly the limit studied by Hocking (1970).

The departure of the angular velocity from rigid-body rotation, $\Gamma/r^2 - 1$, at $\tau = \tau_F$ is an estimate of the importance of nonlinear effects in the filling process. The maximal deviation available from the foregoing solution is at $r = R_i(\tau_F)$, and this particular value of Γ/r^2-1 is defined as the Rossby number ϵ of the filling process under consideration. This value of ϵ is indeed representative for the entire flow field. Upon elimination of $\dot{q}$ from (2.8) and (2.9), we obtain

$$\epsilon = \frac{\Gamma[R_i(\tau_F), \tau_F]}{R_i^2(\tau_F)} - 1 = \frac{b^2 f(\tau_F)}{a^2 - b^2 f(\tau_F)}, \tag{2.16}$$

where

$$b^2 = a^2 - r_I^2, \quad f(\tau_F) = \frac{1-e^{-2\tau_F}}{2\tau_F}.$$

Hence

$$\left.\begin{aligned} \epsilon &\approx \left(\frac{a}{r_I}\right)^2 - 1 \quad (\tau_F \ll 1), \\ \epsilon &\approx \frac{1}{2\tau_F}\left[1 - \left(\frac{r_I}{a}\right)^2\right] \quad (\tau_F \gtrsim 1). \end{aligned}\right\} \tag{2.17}$$

It is now obvious that ϵ is small only for $\tau_F \gg 1$ (we exclude the filling of a narrow gap, $a - r_I \ll 1$, from this analysis). Consequently, the rapid and moderate filling processes are dominated by nonlinear effects, while the slow filling is controlled by the Coriolis (linear) terms. Since for moderate filling times both nonlinear and Ekman-layer contributions are important, this is the most difficult case for analysis.

We shall now focus attention on small-ϵ flows. The first implication, in view of (2.17), is $\tau_F = O(\epsilon^{-1})$. This leads to the interesting conclusion that the flow field, which varies on the spin-up timescale $\tau \sim 1$, is quasi-steady during the filling process. Secondly, elimination of τ_F from (2.14) and (2.17) gives $\dot{q} = O(\epsilon)$. Using this result in (2.12), we find that, to leading order in ϵ, the fluid in the interior core (region I in figure 2) is not displaced during the filling. Moreover, the volume of fluid added to the outer periphery of this core, in region B, is also $O(\epsilon)$. This indicates that the new fluid is entirely transported by the Ekman layers to the radial position r_F of the inner front, where it spreads axially and is added to the core from the *inside*. The details of this flow are discussed in §3.

3. The slow-filling solution

Since the motion is quasi-steady on the spin-up scale, its main part is described by the well-known solution of steady source–sink flow. Thus, referring to figure 4, $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$ vertical shear layers are formed in the source region at $r = 1$, the entire mass flux is transported by *non-divergent* Ekman layers and the azimuthal velocity in the inviscid core is a potential vortex in a system rotating with Ω^* (Hide 1968; Greenspan 1968). If the Rossby number is larger than $E^{\frac{1}{4}}$ the thickness of the layer at $r = 1$ increases, but the motion in the core is unaffected. However, the 'sink' of the abovementioned solution is replaced here by the thin viscous region of the moving front, whose structure is considered now.

The previous scaling for lengths and time is retained and u, v, w are the radial, azimuthal and axial velocity components in the rotating frame scaled by $\epsilon\Omega^* r_0^*$, where, as suggested by (2.14) and (2.17),

$$\epsilon = \dot{q} = \frac{\dot{Q}^*}{2\pi\Omega^* r_0^{*3} E^{\frac{1}{2}}}. \tag{3.1}$$

Note that the scaled volume flux $-\dot{Q}$ is equal to $E^{\frac{1}{2}}$ in this case.

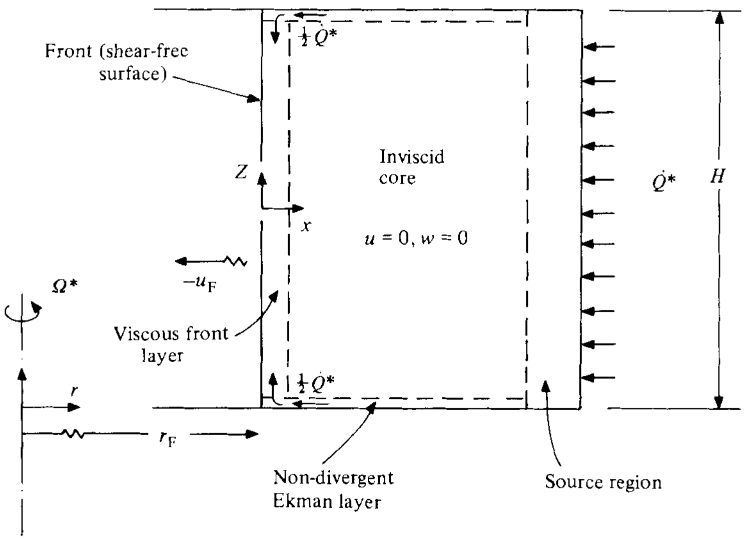

FIGURE 4. Schematic view of the quasi-steady slow-filling configuration.

The analysis is simplified with the transformation

$$x = r - r_F, \quad \mathscr{u} = u - u_F, \tag{3.2}$$

where r_F and u_F are the radial position and velocity of the front. The former variable is given by (2.11), and consequently $u_F = -E^{\frac{1}{2}}/r_F H$.

The equations of motion for the thin viscous layer at the interface are

$$\frac{\partial \mathscr{u}}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{3.3}$$

$$-2v = -\frac{\partial p}{\partial x}, \tag{3.4}$$

$$2\mathscr{u} = E\frac{\partial^2 y}{\partial x^2}, \tag{3.5}$$

$$0 = -\frac{\partial p}{\partial z} + E\frac{\partial^2 w}{\partial x^2}, \tag{3.6}$$

where p is the reduced pressure. The front is considered to be a shear-free cylindrical surface, and this implies the boundary conditions

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0, \quad \mathscr{u} = 0 \quad \text{at} \quad x = 0. \tag{3.7}$$

The solution of (3.3)–(3.6) is also required to match the inviscid interior where

$$v = \frac{1}{r_F + x}, \quad \mathscr{u} = -u_F, \quad w = 0. \tag{3.8}$$

In addition, the Ekman compatibility condition, similar to that given by (2.3), is to be satisfied at $z = 0, H$.

It can be anticipated that the flow involves 'vertical' shear layers. The free surface requires a weak $E^{\frac{1}{4}}$ layer, where the v-correction is $O(E^{\frac{1}{4}})$, but the more-significant structure is the $E^{\frac{1}{3}}$ layer.

To leading order. the solution in these layers is

$$\psi = \frac{E^{\frac{1}{2}}}{r_{\mathrm{F}}}\left[Z\left(1-\frac{\delta}{r_{\mathrm{F}}}\mathrm{e}^{-x/\delta}\right)+\frac{2}{\pi\sqrt{3}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}G_n(x)\sin 2n\pi Z\right], \tag{3.9}$$

$$v = \frac{1}{r_{\mathrm{F}}+x}-\frac{\delta}{r_{\mathrm{F}}^2}\mathrm{e}^{-x/\delta}+\frac{E^{\frac{1}{6}}}{r_{\mathrm{F}}}\left(\frac{4\pi}{H}\right)^{\frac{1}{3}}\frac{2}{\sqrt{3}\,\pi}\Sigma\frac{(-1)^n}{n^{\frac{2}{3}}}F_n(x)\cos 2n\pi Z, \tag{3.10}$$

where

$$\left.\begin{aligned}
u &= \frac{\partial\psi}{\partial z}, \quad w = -\frac{\partial\psi}{\partial x},\\
Z &= \frac{z}{H}-\tfrac{1}{2},\\
F_n(x) &= \mathrm{e}^{-\frac{1}{2}\beta_n x}\sin\left(\tfrac{1}{2}\sqrt{3}\,\beta_n x-\tfrac{2}{3}\pi\right),\\
G_n(x) &= \mathrm{e}^{-\frac{1}{2}\beta_n x}\sin\left(\tfrac{1}{2}\sqrt{3}\,\beta_n x+\tfrac{2}{3}\pi\right),\\
\beta_n &= \left(\frac{4n\pi}{EH}\right)^{\frac{1}{3}},\\
\delta &= \left(\tfrac{1}{2}H\right)^{\frac{1}{2}}E^{\frac{1}{4}}.
\end{aligned}\right\} \tag{3.11}$$

Since the solution of this moving viscous layer is independent of the details in the source region, it can be directly applied for different filling methods. Moreover, it is obvious that time variation of $\dot{Q}^*$ on the filling timescale is implicitly incorporated in these results.

4. Concluding remarks

Three cases of radial filling of a rotating axisymmetric container were distinguished on the basis of the parameter τ_{F}, which is essentially the ratio of the filling and the spin-up times. In 'rapid' filling ($\tau_{\mathrm{F}} \ll 1$) the Ekman-layers contribution is unimportant, and rings of new fluid accumulate at the outer periphery of the fluid already present in the container, pushing it inwards. This case was investigated by Hocking (1970). In 'moderate' filling ($\tau_{\mathrm{F}} = O(1)$) a significant part of the entering fluid is transported to the other side of the core region by the Ekman layers while the remainder accumulates as before. This core is convected inwards as a bulk. Although the adjacent Ekman layers are non-divergent, this motion is similar in many aspects to the spin-up process. The motion of the new fluid transported ahead of this bulk remains open to future investigation.

The 'slow' filling ($\tau_{\mathrm{F}} \gg 1$) is the only case in which the Rossby number is small. The flow is controlled by the Ekman layers and is quasi-steady (on the spin-up scale). The entering fluid is transported by the Ekman layers to the front region, where it spreads through an $E^{\frac{1}{3}}$ layer imbedded in a weak $E^{\frac{1}{4}}$ layer. In this case, filling is an inside-out process.

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Appendix. The viscous layer at $R_0(\tau)$

If the radial shear term in the azimuthal momentum equation (2.1) is not neglected, then the equation corresponding to (2.5) is

$$\frac{\partial \Gamma}{\partial t}+2[(\Gamma-\eta)-\dot{q}]\frac{\partial \Gamma}{\partial \eta} = 4HE^{\frac{1}{2}}\eta\frac{\partial^2 \Gamma}{\partial \eta^2}, \tag{A 1}$$

where

$$\eta = r^2.$$

Since the viscous layer develops around

$$\eta_0(\tau) = R_0^2(\tau) = (1-\dot{q})+\dot{q}\,\mathrm{e}^{-2\tau}, \tag{A 2}$$

we define the new variables

$$\zeta = \frac{1}{\alpha}[\eta-\eta_0(\tau)], \tag{A 3}$$

where

$$\alpha = (2H)^{\frac{1}{2}}E^{\frac{1}{4}}, \tag{A 4}$$

$$\theta = 1-\mathrm{e}^{-2\tau}, \tag{A 5}$$

such that $\zeta = 0$ is the centre of the viscous region and θ varies from 0 to 1 as τ changes from 0 to ∞. Moreover, because $\Gamma = 1$ at $\eta_0(\tau)$, it is convenient to introduce

$$\gamma = \Gamma-1. \tag{A 6}$$

Substitution and rearrangement results in

$$\gamma_\theta+\frac{1}{\alpha(1-\theta)}(\gamma-\alpha\zeta)\,\gamma_\zeta = [1-\dot{q}\theta+\alpha\zeta]\frac{1}{1-\theta}\gamma_{\zeta\zeta}. \tag{A 7}$$

The matching to the inviscid solution (2.8) provides the boundary conditions

$$\gamma = \begin{cases} 0 & (\zeta > \delta/\alpha), \\ \alpha\zeta & (\zeta < -\delta/\alpha), \end{cases} \tag{A 8}$$

where 2δ is the thickness of the viscous region. Because δ develops initially by diffusion, we take $\delta = 0$ at $\theta = 0$.

Since $\alpha \ll 1$ we can neglect $\alpha\zeta$ in the right-hand-side term of (A 7). In addition, we anticipate that the second term of the left-hand side is unimportant for small θ. Discarding it, we obtain

$$\gamma_T = \gamma_{\zeta\zeta}, \tag{A 9}$$

where

$$T = \int_0^\theta \frac{1-\dot{q}\theta}{1-\theta}\,\mathrm{d}\theta = \dot{q}\theta-(1-\dot{q})\ln(1-\theta). \tag{A 10}$$

The solution of this equation, subject to the initial conditions (A 8) with $\delta = 0$, is

$$\gamma = \alpha T^{\frac{1}{2}}[\rho\,\mathrm{erfc}\,(\rho)-\frac{1}{\pi^{\frac{1}{2}}}\exp(-\rho^2)], \tag{A 11}$$

where

$$\rho = \frac{\zeta}{2T^{\frac{1}{2}}}, \quad \mathrm{erfc}\,\rho = \frac{2}{\pi^{\frac{1}{2}}}\int_\rho^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x. \tag{A 12}$$

This solution indicates that $\gamma = O(\alpha T^{\frac{1}{2}})$, $\delta = O(\alpha T^{\frac{1}{2}})$ and the relative magnitude of the neglected nonlinear term is $O(T)$. Consequently, the solution (A 11) is valid for small

T (or θ). For larger values of θ a numerical solution of (A 7), where (A 11) serves as a smoothed initial condition, can be attempted. It is interesting to note that for the particular case $\dot{q} = 1$ (A 1) can be easily reduced to one of Burgers' type. In this case, $\eta_0 = e^{-2\tau}$, and we define the variables ϕ and χ:

$$\phi = (\eta - \eta_0)\, e^{2\tau}, \quad \chi = e^{2\tau} - 1. \tag{A 13}$$

Upon substitution in (A 1) and using (A 6), we obtain

$$\gamma_\chi + \gamma\gamma_\phi = \alpha^2 \gamma_{\phi\phi}(1+\phi). \tag{A 14}$$

If $\phi (= O(\alpha))$ is neglected in the last term, the remaining equation can be solved (see e.g. Carrier & Pearson 1976) subject to

$$\gamma = \begin{cases} 0 & (\phi \geqslant 0), \\ \phi & (\phi < 0). \end{cases} \tag{A 15}$$

A similar Burgers' equation (but different boundary condition) governs the viscous layer attached to the discontinuity point in nonlinear spin-up from rest problem (see Venezian 1970; Weidman 1975; Hyun *et al.* 1983).

However, since the filling is completed here in $\tau_F < 0.5$, which corresponds to $\theta < 0.63$, the simple solution (A 11) seems a fairly accurate reproduction of this viscous layer during the entire filling process when $\dot{q} = 1$.

The foregoing analysis neglects the interaction between the viscous layer and the outer wall. Its validity is consequently restricted to $\delta \ll 1 - R_0$, which implies $\dot{q} \gg \alpha$ or $\epsilon \gg H^{\frac{1}{2}}E^{\frac{1}{4}}$, where ϵ is the Rossby number.

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